The "Continuous Monotone Convexity (CMC)" Property and its Implications to Statistical Modeling

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In a previous post we have discussed why the Box-Cox (BC) transformation is so effective when applied to a response variable in linear regression analysis. We concluded that the inverse BC transformation is effective due to its Continuous Monotone Convexity (CMC) property, which renders a trio of fundamental monotone convex functions, "linear-power-exponential", into points on the *continuous spectrum of monotone convexity*. We have likened this to the scientific realization, not long ago, that colors, formerly treated as separate entities, are in fact mere points on the continuous spectrum of electromagnetic radiation.

In this article, we elaborate further on the CMC property and its implications to the state-of-the-art of statistical modeling. We start with the theoretical motivation for addressing the CMC property as a basic platform for statistical modeling.

The statistics discipline is awash with models. Broadly speaking, these models may be partitioned into three categories (the classification herewith is not exhaustive but probably encompasses the majority of current types of statistical models): Models of *random variation* (also denoted "statistical distributions"), models of *systematic variation* (where a large proportion of a response variability is accounted for by variability in covariates) and *time-series* models. Henceforth we will confine our discussion to the first two categories. Inspecting models belonging to these categories, as reflected in the scientific and statistical literature, one realizes that monotone convexity plays a major role, either directly or indirectly. For example, a statistical distribution may be uniquely modeled in terms of its *normal-based* quantile function (the response quantile expressed in terms of the standard normal quantile). In this, admittedly non-conventional, mode of modeling random variation, monotone convexity plays a central role (Shore, 2013a).

Despite the abundance of statistical models that one can find in the statistics literature, serious efforts at unifying these models have rarely been reported. This state-of-the-art stands in stark contrast to efforts at unification conducted in other branches of science. Perhaps the most well-known of these is in modern physics, where a *Theory of Everything* is continuously being attempted in order to unify gravitation with the other already-unified four fundamental forces of nature (the electric force, the magnetic force, the weak nuclear force and the strong nuclear force). A most recent expression of these efforts are super-string-theories. Thus, the "objects of enquiry" of modern physics, the five forces of nature, have been mostly unified to provide a single model comprising the individual forces as "special cases". Other branches of science, like modern biology, have pursued a similar approach with varying degrees of success. Regrettably, no similar attempts are observed in the current statistics literature.

This article reviews my recent attempts (published over the last fifteen years or so) to employ the CMC property as basic principle (or platform) for unifying separate statistical models under a shared "umbrella" model. As noted in a previous post, the inverse BC transformation has already done so (namely, using the CMC principle to unify models), however only partially.

To realize the prevalence of the basic trio of functions, the "linear-power-exponential" trio, in various disciplines of science and technology, it might be useful to review their occurrence rates in the relevant literature. Doing so, one may realize that the three fundamental functions that comprise the inverse BC transformation are indeed widely embedded in quantitative models to achieve varying levels of monotone convexity. An extreme example is Gompertz growth model, which uses an exponential-exponential relationship. However, models of more moderate levels of monotone convexity, like those of exponential-power relationships, are quite common. In Shore (2004) we have surveyed these models. Figure 1 shows some well-known examples. We realize that the inverse BC transformation represents just the first three levels in a hierarchy of models, all based on repeated use of the fundamental "linear-power-exponential" relationships. As one ascends this hierarchy, models with stronger monotone convexity are encountered. This hierarchy had been denoted by us "The Ladder of Monotone Convex Functions" (Shore, 2005). The Ladder is demonstrated in Figure 2.

As the inverse BC transformation unifies the first three steps on the Ladder, can all models of the Ladder be similarly unified (namely, expressed via a single model)?

Some fifteen year ago I have started developing a new modeling approach, denoted Response Modeling Methodology (RMM) (Shore, 2005, 2011, 2012 and references therein). The *basic* RMM model represents the first four steps of the Ladder (Figure 2), thus delivering an extra "step" (over the inverse BC transformation). At the "cost" of additional parameters, introduced into the RMM basic model, one can keep "ascending" the Ladder to obtain models with higher levels of monotone convexity (refer to Figure 3). In fact, for a single additional iteration in the use of the "linear-power-exponential" trio, two additional parameters need to be introduced into the model. Needless to say that if the final shape of a model is determined solely by the values of its parameters, rather than being specified in advance, data-based estimation of these parameters would deliver more adequate models (better goodness-of-fit) than those that are pre-specified (let alone those erroneously pre-specified, namely, mis-specified). Furthermore, cumulative experience with RMM modeling has shown that, due to the small number of parameters of RMM models, general measures of the quality of estimated models (like AIC or BIC) rarely indicate over-fit.

To demonstrate how RMM delivers representation to all models of the Ladder, we remove the model's two random error terms (part and parcel of the RMM model) and address the underlying deterministic core of the model (with Y as the response and η is the linear predictor, a linear combination of covariates):

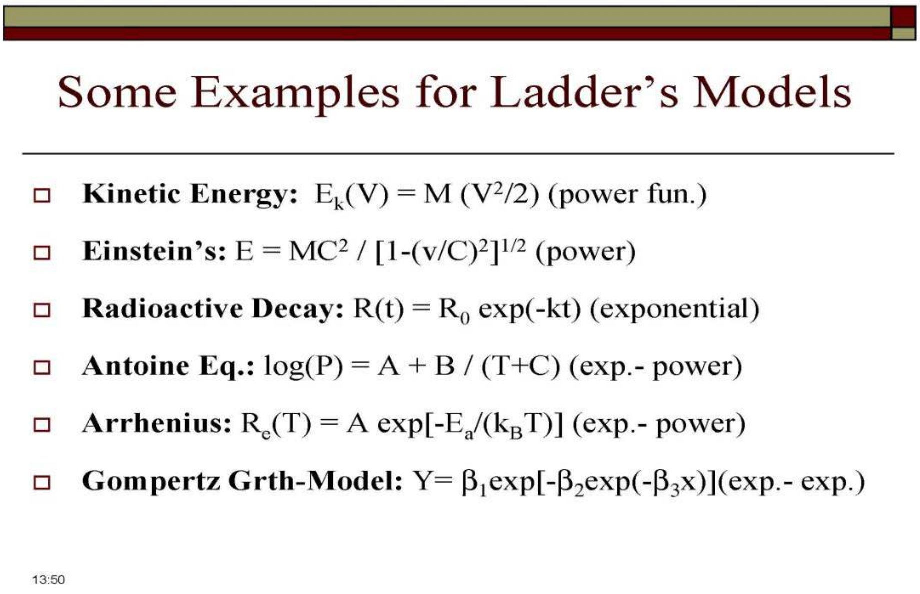

Figure 1. Some known examples of models belonging to the "Ladder of monotone convex functions"

(Image taken from a PowerPoint presentation on RMM).

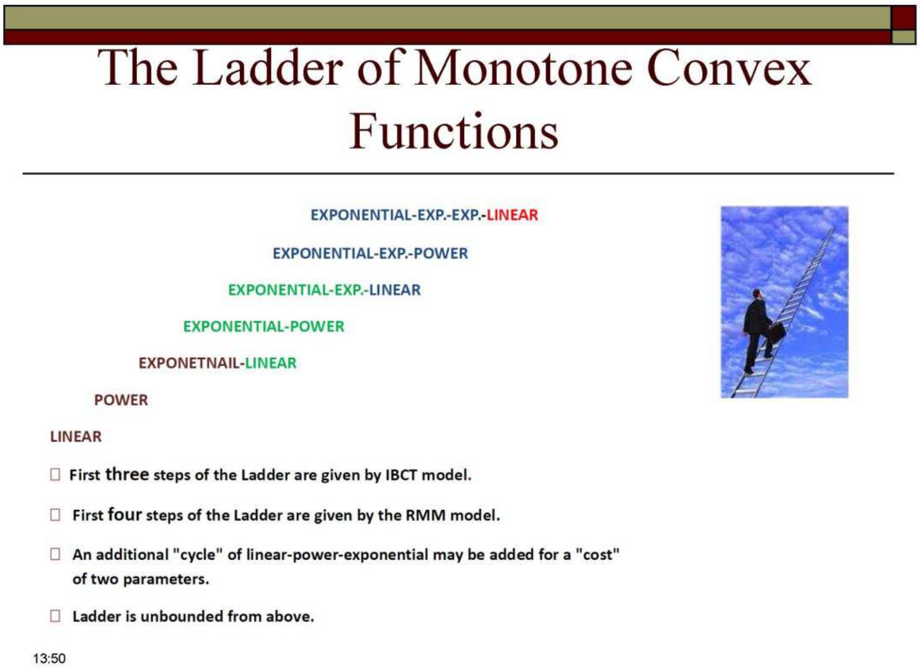

Figure 2. The "Ladder of monotone convex functions"

(Image taken from a PowerPoint presentation on RMM; IBCT is Inverse BC Transformation).

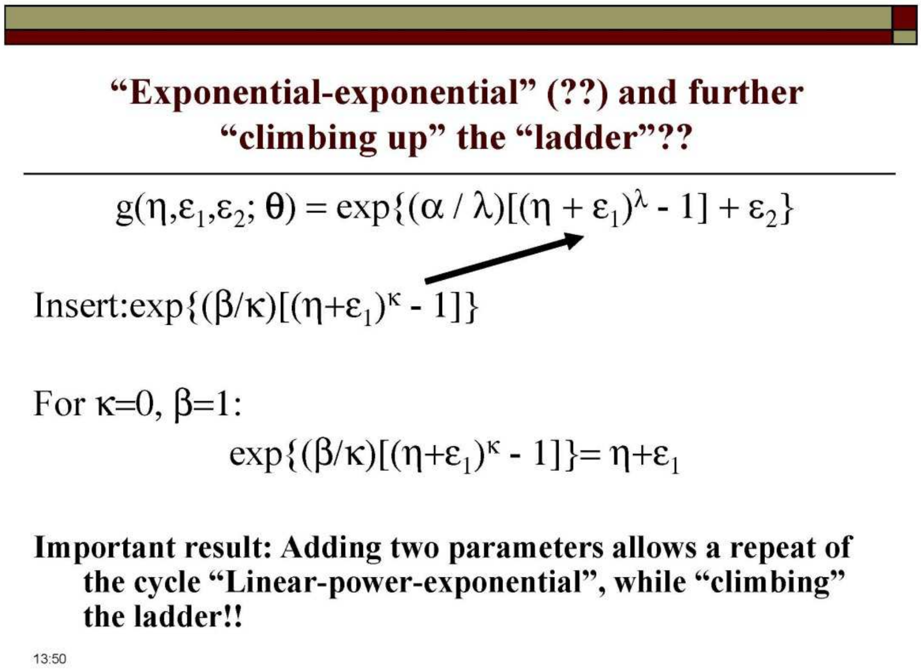

Figure 3. Ascending the Ladder by introducing two additional parameters, β and κ;

ε_1 and ε_2 are two possibly correlated normal random errors

(Image taken from a PowerPoint presentation on RMM)

$$W = \log(Y) = \mu + \frac{\alpha}{\lambda}[(\eta)^{\lambda} - 1],$$

where (α, λ, μ) are parameters. It is easy to realize that apart from the parameters contained in the linear predictor (η), this model has only two additional parameters (namely, α can be absorbed in η and in the location parameter).

By properly selecting the values of the two parameters, α and λ, the first four models of the Ladder are obtained:

* linear (α=1, λ=0); * power ($\alpha \neq 1$, λ=0);

* exponential–linear (λ=1); * exponential–power ($\lambda \neq 0,1$);

Introducing two new parameters, β and κ (as demonstrated in Figure 3), one may climb up two steps further on the Ladder to obtain relationships with stronger convexity:

* exponential-exponential-linear; * exponential-exponential-power;

This procedure may be repeated, thereby rendering the number of "steps" on the Ladder unbounded from above. In other words: RMM can deliver, in a continuous fashion, any desirable degree of monotone convexity.

How well can an RMM model, based on the CMC property, represent current (published) statistical models or stochastic scientific models, which have been derived from theoretical considerations based on well-established theories?

In fact, quite well. This has been demonstrated in numerous published articles (and some that are currently under review), for example: Shore (2013b, 2013c); Shore *et al.* (2013), Benson-Karhi *et al.*(2013) and references therein. In a future post, we will present the RMM model in a more comprehensive fashion and then relate also to the empirical cumulative evidence for the "generality" of the RMM approach in modeling monotone convex relationships.

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